MATH-SHU 350

Recitation 4: Convergence Theorems

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Exercise 1. *Show that, if X and Y are random variables in $(\Omega, \mathcal{F}, \mathbb{P})$, then $X + Y$ is also a random variable by checking*

$$\forall x \in \mathbb{R}, \qquad \{X + Y < x\} \in \mathcal{F}.$$

Exercise 2. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $f : \Omega \to [0, \infty]$ be a measurable function. Show that*

$$\lim_{n\to\infty} \int_{\Omega} n \log\left(1 + \frac{f}{n}\right) d\mu = \int_{\Omega} f d\mu.$$

Exercise 3. *For a sequence of random variables $(X_n)_{n\geqslant 1}$, show that if the condition "X_n is non-negative" is not satisfied, we cannot apply Fatou's lemma.*

Exercise 4 (Scheffé's lemma)**.** *Let $(X_n)_{n\geqslant 1}$ be positive random variables and $X_n \xrightarrow{a.s.} X$. We suppose moreover $\mathbb{E}[X_n] = 1$ for all $n \in \mathbb{N}_+$. Prove that*

$$\mathbb{E}[X] = 1 \Longleftrightarrow X_n \xrightarrow{L^1} X.$$

Exercise 5 (Ond-sided Chebyshev bound)**.** *Suppose that $\mathbb{E}[X] = 0$, $\mathrm{Var}[X] = \sigma^2$. Then for any $a > 0$, prove that $\mathbb{P}[X \geqslant a] \leqslant \frac{\sigma^2}{a^2+\sigma^2}$. Moreover, for a fixed $a > 0$, there exists random variable such that "=" holds.*